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$$\vec{r} = R\cos(\lambda t)\hat{\imath} + R\sin(\lambda t)\hat{\jmath}\ ,\quad r = R$$
$$\vec{v} = -R\lambda\sin(\lambda t)\hat{\imath} + R\lambda\cos(\lambda t)\hat{\jmath}\ ,\quad v = \lambda R$$
$$\vec{a} = -R\lambda^2\left[\cos(\lambda t)\hat{\imath} + \sin(\lambda t)\hat{\jmath}\right] = -\lambda^2\vec{r}\ ,\quad a = -\lambda^2 R$$

a) $F(\vec{r}) = -k\vec{r} = m\vec{a}$

$$\therefore\ -k\vec{r} = -m\lambda^2\vec{r} \quad \text{OR} \quad (-k + m\lambda^2)\vec{r} = \vec{0}$$
$$-k + m\lambda^2 = 0 \Rightarrow \lambda = \sqrt{k/m}\ ,\ \text{so}\quad v = R\lambda = \boxed{R\sqrt{k/m}}$$

b) $F(\vec{r}) = -\dfrac{GMm}{r^2}\dfrac{\vec{r}}{r} = m\vec{a}$

$$\therefore\ -\frac{GMm}{R^3}\vec{r} = -m\lambda^2\vec{r} \quad \text{OR} \quad \left(-\frac{GMm}{R^3} + m\lambda^2\right)\vec{r} = \vec{0}$$
$$-\frac{GM}{R^3} + \lambda^2 = 0 \Rightarrow \lambda = \sqrt{\frac{GM}{R^3}}\ ,\ \text{so}\quad \boxed{v = \sqrt{\frac{GM}{R}}}$$

c) $F(\vec{r}) = -r^n\dfrac{\vec{r}}{r} = m\vec{a}$

$$\therefore\ -R^{n-1}\vec{r} = -m\lambda^2\vec{r}$$
$$-R^{n-1} + m\lambda^2 = 0 \Rightarrow \lambda = \sqrt{\frac{R^{n-1}}{m}}\ ,\ \text{so}\quad \boxed{v = \sqrt{\frac{R^{n+1}}{m}}}$$

d) $F(\vec{r}) = -F(r)\dfrac{\vec{r}}{r} = m\vec{a}$

$$-F(r)\frac{\vec{r}}{R} = -m\lambda^2\vec{r}$$
$$-\frac{F(r)}{R} + m\lambda^2 = 0 \Rightarrow \lambda = \sqrt{\frac{F(r)}{mR}}\ ,\ \text{so}\quad \boxed{v = \sqrt{\frac{RF(R)}{m}}}$$

e) To have the same speed v, we need $\dfrac{RF(R)}{m}$ to be constant, for any R.

$\therefore$ $\boxed{F(r) = k/r}$ satisfies this condition, assuming constant mass, m.

$$\Rightarrow v = \sqrt{k/m}$$